FP3 June 11 M.A. Perime 2

1. The curve C has equation $y = 2x^3$, $0 \leqslant x \leqslant 2$.

The curve C is rotated through 2π radians about the x-axis.

Using calculus, find the area of the surface generated, giving your answer to 3 significant figures.

(5)

$$y = 2x^3 \Rightarrow \frac{dy}{dx} = 6x^2$$

$$\text{Area} = 2\pi \int_0^2 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\therefore \text{Area} = 2\pi \int_0^2 2x^3\sqrt{1+36x^4}\,dx$$

$$= 4\pi \int_0^2 x^3 (1+36x^4)^{1/2}\,dx$$

$$\int f'(x) f(x)^n\,dx \quad \cancel{= 4\pi [} \quad \cancel{= 4\pi}$$

$$\cancel{= 4\pi} = \frac{\pi}{36}\int_0^2 144x^3 (1+36x^4)^{1/2}\,dx$$

$$= \frac{\pi}{36}\left[\frac{(1+36x^4)^{3/2}}{3/2}\right]_0^2 = \frac{\pi}{36}\left[\frac{577^{3/2}}{3/2} - \frac{1}{3/2}\right]$$

$$= 806 \text{ units}^2$$

(3sf)

$$\cancel{= \frac{\pi}{36}\left(\frac{73^{3/2}}{3/2} - \frac{1}{3/2}\right)}$$

$$\cancel{= 36.2 \text{ units}^2}$$

2. (a) Given that $y = x \arcsin x$, $0 \leqslant x \leqslant 1$, find

(i) an expression for $\frac{dy}{dx}$,

(ii) the exact value of $\frac{dy}{dx}$ when $x = \frac{1}{2}$.

(3)

(b) Given that $y = \arctan(3e^{2x})$, show that

$$\frac{dy}{dx} = \frac{3}{5\cosh 2x + 4\sinh 2x}$$

(5)

2(a) $y = x\arcsin x$

(i) Product rule:

$$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \arcsin x$$

(ii) $$\left(\frac{dy}{dx}\right)_{x=1/2} = \frac{1/2}{\sqrt{1-1/4}} + \arcsin\frac{1}{2}$$

$$= \frac{\sqrt{3}}{3} + \frac{\pi}{6} = \frac{2\sqrt{3}+\pi}{6}$$

(b) $y = \arctan(3e^{2x})$ ← ~~use chain rule~~

$\cancel{\frac{dy}{dx} = 6e^{2x}}$ $\tan y = 3e^{2x}$

$y = \arctan(3e^{2x})$

$\therefore \tan y = 3e^{2x}$

$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2}$

$\therefore \frac{dy}{dx} \sec^2 y = 6e^{2x}$

$1 + t^2 = \sec^2$

$\therefore \frac{dy}{dx}(\tan^2 y + 1) = 6e^{2x}$

$\therefore \frac{dy}{dx} = \frac{6e^{2x}}{\tan^2 y + 1} = \frac{6e^{2x}}{9e^{4x} + 1}$

Let $\therefore \frac{dy}{dx} = \frac{\cancel{6}\, 6e^{2x}}{9e^{4x} + 1} \times \frac{\frac{1}{2e^{2x}}}{\frac{1}{2e^{2x}}}$

$= \frac{3}{\frac{9}{2}e^{2x} + \frac{1}{2}e^{-2x}} = LHS$

Let $\frac{dy}{dx} = \frac{3}{\frac{9}{2}e^{2x} + \frac{1}{2}e^{-2x}} = LHS$

$$RHS = \frac{3}{5\cosh 2x + 4\sinh 2x} = \frac{3}{5.\frac{e^{2x}+e^{-2x}}{2} + 4.\frac{e^{2x}-e^{-2x}}{2}}$$

$$= \frac{3}{\frac{5e^{2x}+5e^{-2x}+4e^{2x}-4e^{-2x}}{2}}$$

$$= \frac{3}{\frac{9e^{2x}+e^{-2x}}{2}}$$

$$= \frac{3}{\frac{9}{2}e^{2x} + \frac{1}{2}e^{-2x}}$$

$= LHS$

$$\therefore \frac{dy}{dx} = \frac{3}{5\cosh 2x + 4\sinh 2x}$$

as required

3. Show that

(a) $\displaystyle\int_5^8 \frac{1}{x^2-10x+34}\,dx = k\pi$, giving the value of the fraction k, **(5)**

(b) $\displaystyle\int_5^8 \frac{1}{\sqrt{(x^2-10x+34)}}\,dx = \ln(A+\sqrt{n})$, giving the values of the integers A and n. **(4)**

3(a). $$\int_5^8 \frac{1}{x^2-10x+34}\,dx = \int_5^8 \frac{1}{(x-5)^2+9}$$

$$= \left[\frac{1}{3}\arctan\frac{x-5}{3}\right]_5^8$$

$$= \frac{1}{3}\arctan 1 - \frac{1}{3}\arctan 0$$

$$= \frac{1}{12}\pi \qquad k = \frac{1}{12}$$

(b) $$\int_5^8 \frac{1}{\sqrt{(x-5)^2+9}}\,dx = \left[\operatorname{arsinh}\frac{x-5}{3}\right]_5^8$$

$$= \operatorname{arsinh} 1 - \operatorname{arsinh} 0$$

$$= \ln(1+\sqrt{2}) - \ln(0+\sqrt{1})$$

$$= \ln(1+\sqrt{2}) \qquad A = 1 \quad n = 2$$

4.

$$I_n = \int_1^e x^2 (\ln x)^n \, dx, \quad n \geqslant 0$$

(a) Prove that, for $n \geqslant 1$,

$$I_n = \frac{e^3}{3} - \frac{n}{3} I_{n-1}$$

(4)

(b) Find the exact value of I_3.

(4)

(a). $I_n = \int_1^e x^2 (\ln x)^n \, dx$

Let $u = (\ln x)^n \qquad u' = \frac{n(\ln x)^{n-1}}{x}$

$v' = x^2 \qquad v = \frac{1}{3} x^3$

$\therefore I_n = \left[\frac{1}{3} x^3 (\ln x)^n\right]_1^e - \frac{1}{3} n \int_1^e \frac{x^3 (\ln x)^{n-1}}{x} \, dx$

$= \frac{1}{3} e^3 - \frac{1}{3} n \int_1^e \frac{x^3 (\ln x)^{n-1}}{x} \, dx$

$= \frac{e^3}{3} - \frac{n}{3} \int_1^e x^2 (\ln x)^{n-1} \, dx$

$= \frac{e^3}{3} - \frac{n}{3} I_{n-1}$ as required.

$\frac{x^2}{x^3}$

Question 4 continued

(b) $I_3 = \frac{e^3}{3} - \cancel{3}\, I_2$

$= \frac{e^3}{3} - \left(\frac{e^3}{3} - \frac{2}{3} I_1\right)$

$\cancel{= \frac{e^3}{2} + \frac{2}{2}}$

$= \frac{2}{3} I_1 = \frac{2}{3}\int_1^e x^2 \ln x \, dx$

~~$u = x$~~ $u = \ln x \quad u' = \frac{1}{x}$

$v' = x^2 \quad v = \frac{1}{3}x^3$

$\cancel{= \frac{2}{3}\int}$ $= \frac{2}{3}\left(\left[\frac{x^3}{3}\ln x\right]_1^e - \frac{1}{3}\int_1^e x^2 dx\right)$

$\cancel{= 3}$ $= \frac{2}{3}\left(\frac{e^3}{3} - \frac{1}{3}\left[\frac{1}{3}x^3\right]_1^e\right)$

$= \frac{2}{3}\left(\frac{e^3}{3} - \frac{1}{3}\left(\frac{e^3}{3} - \frac{1}{3}\right)\right)$

$= \frac{2}{3}\left(\frac{2}{9}e^3 + \frac{1}{9}\right) = \frac{4}{27}e^3 + \frac{2}{27}$

$= \frac{4e^3+2}{27}$ //

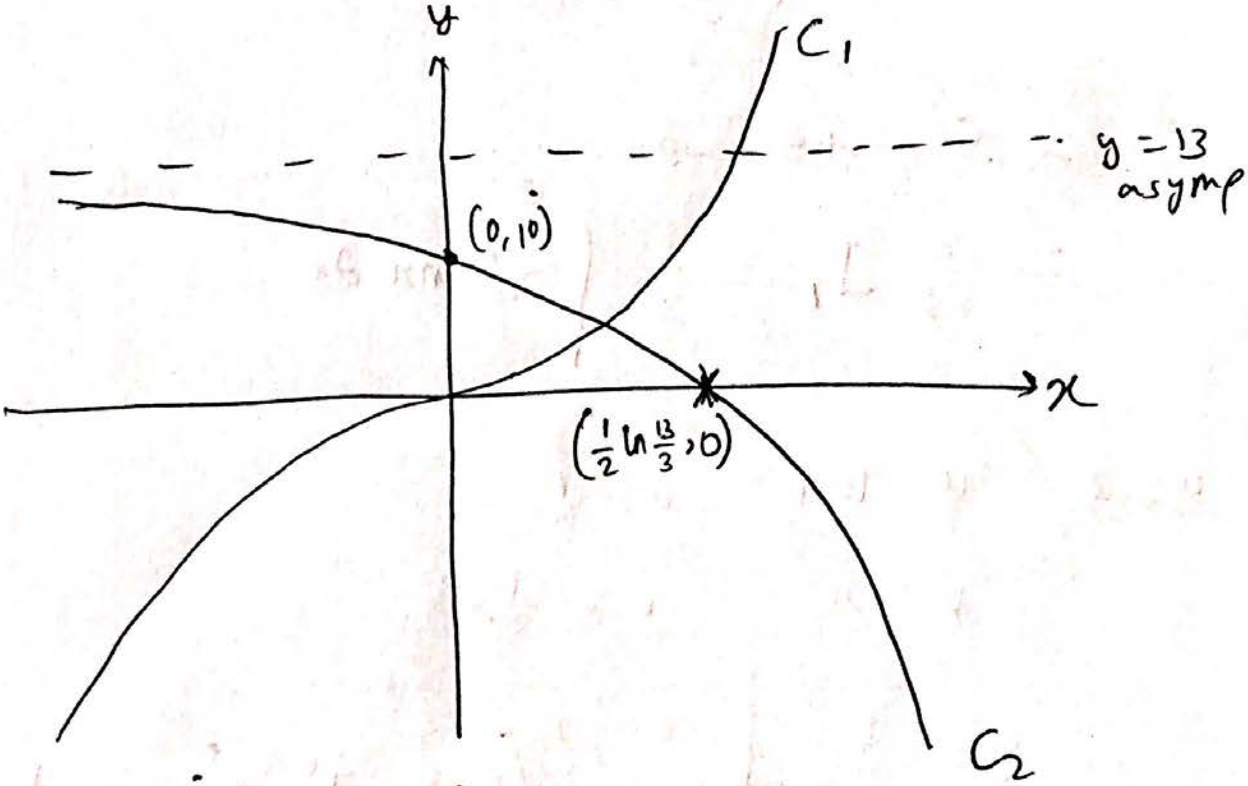
5. The curve C_1 has equation $y = 3\sinh 2x$, and the curve C_2 has equation $y = 13 - 3\mathrm{e}$

(a) Sketch the graph of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes.

(4)

(b) Solve the equation $3\sinh 2x = 13 - 3\mathrm{e}^{2x}$, giving your answer in the form $\frac{1}{2}\ln k$, where k is an integer.

(5)



Question 5 continued

b) $3\sinh 2x = 13 - 3e^{2x}$

$$\frac{3e^{2x} - 3e^{-2x}}{2} = 13 - 3e^{2x}$$

$\therefore \quad 3e^{2x} - 3e^{-2x} = 26 - 6e^{2x}$

$\therefore \quad 3e^{4x} - 3 = 26e^{2x} - 6e^{4x}$

$\therefore \quad 9e^{4x} - 26e^{2x} - 3 = 0$

$\therefore \quad (e^{2x} - 3)(9e^{2x} + 1) = 0$

$\therefore e^{2x} = 3 \Rightarrow 2x = \ln 3 \Rightarrow x = \frac{1}{2}\ln 3$

$e^{2x} \neq -\frac{1}{9}$ ~~$\Rightarrow 4x = \ln$~~

6. The plane P has equation

$$\mathbf{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}3\\2\\2\end{pmatrix}$$

(a) Find a vector perpendicular to the plane P.

(2)

The line l passes through the point A (1, 3, 3) and meets P at (3, 1, 2).

The acute angle between the plane P and the line l is α.

(b) Find α to the nearest degree.

(4)

(c) Find the perpendicular distance from A to the plane P.

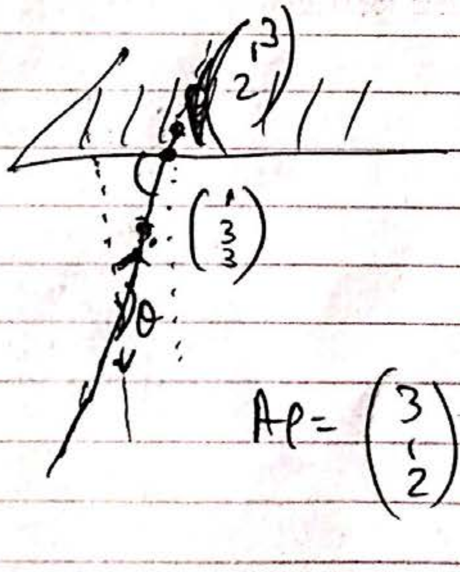
(4)

6(a).

$$\underset{\sim}{n} = \begin{pmatrix}0\\2\\-1\end{pmatrix} \times \begin{pmatrix}3\\2\\2\end{pmatrix} =$$

$$\underset{\sim}{n} = \begin{pmatrix}6\\-3\\-6\end{pmatrix}$$

(b)



$$\underset{\sim}{r} \cdot \begin{pmatrix}6\\-3\\-6\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix} \cdot \begin{pmatrix}6\\-3\\-6\end{pmatrix}$$

$$= 18 - 3 - 12$$

$$\underset{\sim}{r} \cdot \underline{n} = 3$$

$$AP = \begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}1\\3\\3\end{pmatrix} = \begin{pmatrix}2\\-2\\-1\end{pmatrix}$$

Question 6 continued

$$\cos\theta = \frac{\begin{pmatrix}2\\-2\\-1\end{pmatrix}\cdot\begin{pmatrix}6\\-3\\-6\end{pmatrix}}{\sqrt{2^2+2^2+1}+\sqrt{6^2+3^2+6^2}} = \frac{24}{3\times 9}$$

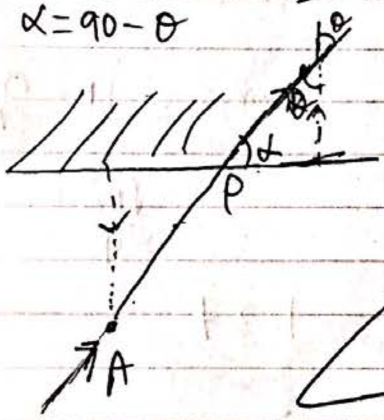
$$= \frac{8}{9}$$

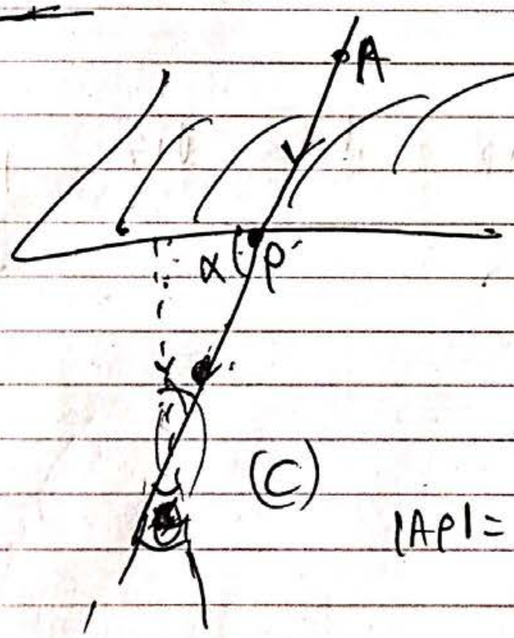
$\Rightarrow \theta = 27.266\ldots$

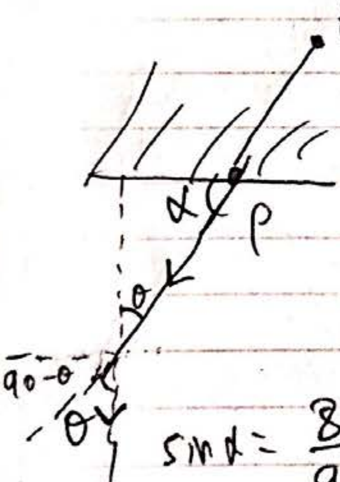
$\cancel{\alpha =}$ $\alpha = 90 - 27°$ (nearest degree)

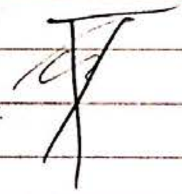
$\alpha = 63°$

$\alpha = 90 - \theta$









(c) $|AP| = \sqrt{2^2+2^2+1^2} = 3$

$\sin\alpha = \frac{8}{9}$

$\therefore$ $\cancel{\sin\alpha =}$ $\cancel{\cos\alpha =}$ $\sin\alpha = \frac{d}{3}$

$\cancel{\therefore d = 3\sin\alpha = 3\times\sqrt{1-\left(\frac{8}{9}\right)^2}}$

$d = 3\sin\alpha = 3\times 8/9 = \frac{8}{3}$

7. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} k & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}, \quad k \neq 1$$

(a) Show that det $\mathbf{M} = 2 - 2k$.

(2)

(b) Find $\mathbf{M}^{-1}$, in terms of k.

(5)

The straight line l_1 is mapped onto the straight line l_2 by the transformation represented by the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}$.

The equation of l_2 is $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$, where $\mathbf{a} = 4\mathbf{i} + \mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

(c) Find a vector equation for the line l_1.

(5)

~~7(a) $\det(M) = k(0-2) + (1+3) + (-2)$~~

~~$= -2k + 4 - 2 = 2 - 2k$~~ as required

7(a) $\det(M) = k\begin{vmatrix} 0 & -1 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$

$= k(0-2) + (1+3) + 1(-2)$

$= -2k + 4 - 2 = 2 - 2k$

(b)

$$C = \begin{pmatrix} +(-2) & -(4) & +(-2) \\ -(1) & +(k-3) & -(3-2k) \\ +1 & -(-k-1) & +(1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -4 & -2 \\ -1 & k-3 & 2k-3 \\ 1 & k+1 & 1 \end{pmatrix}$$

$$\therefore C^T = \begin{pmatrix} -2 & -1 & 1 \\ -4 & k-3 & k+1 \\ -2 & 2k-3 & 1 \end{pmatrix}$$

$$\therefore M^{-1} = \frac{1}{2-2k} \begin{pmatrix} -2 & -1 & 1 \\ -4 & k-3 & k+1 \\ -2 & 2k-3 & 1 \end{pmatrix}$$

(c) $k=2$

$M l_1 = l_2$

$l_1 = M^{-1} l_2$

$$M^{-1} = \frac{-1}{2} \begin{pmatrix} -2 & -1 & 1 \\ -4 & -1 & 3 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 1 & -1 \\ 4 & 1 & -3 \\ 2 & -1 & -1 \end{pmatrix}$$

$$\underset{\sim}{r} \times \underset{\sim}{b} = \underset{\sim}{a} \times \underset{\sim}{b}$$

$$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4+4\lambda \\ 1+\lambda \\ 7+3\lambda \end{pmatrix}$$

$$\therefore \; l_1 = \frac{1}{2} \underset{(3\times3)}{\begin{pmatrix} 2 & 1 & -1 \\ 4 & 1 & -3 \\ 2 & -1 & -1 \end{pmatrix}} \underset{(3\times1)}{\begin{pmatrix} 4+4\lambda \\ 1+\lambda \\ 7+3\lambda \end{pmatrix}}$$

$$= \frac{1}{2} \begin{pmatrix} 8+8\lambda+1+\lambda-7-3\lambda \\ 16+16\lambda+1+\lambda-21-9\lambda \\ 8+8\lambda-1-\lambda-7-3\lambda \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2+6\lambda \\ 8\lambda-4 \\ 4\lambda \end{pmatrix} = \begin{pmatrix} 1+3\lambda \\ 4\lambda-2 \\ 2\lambda \end{pmatrix}$$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

8. The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(a) Use calculus to show that the equation of the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$ may be written in the form

$$xb\cosh\theta - ya\sinh\theta = ab$$

(4)

The line l_1 is the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$, $\theta \neq 0$.
Given that l_1 meets the x-axis at the point P,

(b) find, in terms of a and θ, the coordinates of P.

(2)

The line l_2 is the tangent to H at the point $(a, 0)$.
Given that l_1 and l_2 meet at the point Q,

(c) find, in terms of a, b and θ, the coordinates of Q.

(2)

(d) Show that, as θ varies, the locus of the mid-point of PQ has equation

$$x(4y^2 + b^2) = ab^2$$

(6)

8(a). $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ differentiate :

$\therefore \frac{2}{a^2}x - \frac{2}{b^2}y\frac{dy}{dx} = 0$

$\therefore \frac{1\cancel{2}}{a^2}x = \frac{1\cancel{2}}{b^2}y\frac{dy}{dx}$

$\therefore \frac{b^2}{a^2}\frac{x}{y} = \frac{dy}{dx}$

$\therefore \left(\frac{dy}{dx}\right)_{\substack{a\cosh\theta \\ b\sinh\theta}} = \frac{b^2}{a^2} \cdot \frac{a\cosh\theta}{b\sinh\theta} = \cancel{}$

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Question 8 continued

$= \frac{b}{a}\coth\theta \quad = \left(\frac{\partial y}{\partial x}\right) \quad \frac{a\cosh\theta}{b\sinh\theta}$

$\therefore y - b\sinh\theta = \frac{b}{a}\coth\theta\,(x - a\cosh\theta)$

$\therefore y - b\sinh\theta = \frac{b}{a}\coth\theta\, x - b\,\frac{\cosh^2\theta}{\sinh\theta}$

$\times(a\sinh\theta)\therefore \quad ya\sinh\theta - ab\sinh^2\theta = xb\cosh\theta - ab\cosh^2\theta$

$\therefore \cancel{ya\sinh\theta - xb\cosh\theta}$

$\therefore xb\cosh\theta - ya\sinh\theta = ab\cosh^2\theta - \cancel{ab}\sinh^2\theta$

$\therefore \cancel{xb\cosh} \quad xb\cosh\theta - ya\sinh\theta = ab(\cosh^2\theta - \sinh^2\theta)$

$\therefore xb\cosh\theta - ya\sinh\theta = ab$

~~(b) at P, $y=0 \Rightarrow b\sinh\theta = 0$~~

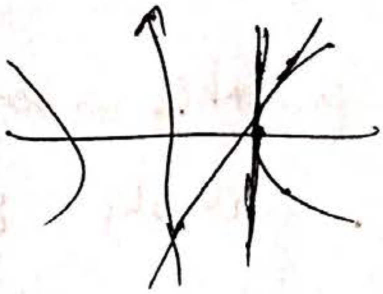
(b) $y=0 \Rightarrow x\cancel{b}\cosh\theta = a\cancel{b}$

$$\Rightarrow x = \frac{a}{\cosh\theta}$$

$$\therefore P\left(\frac{a}{\cosh\theta}, 0\right)$$

(c)

$\frac{a^2}{a^2} \cdot 1 - \frac{y^2}{b^2} = 1$



l_2 has eqn: $x=a$

$x=a \Rightarrow abcosh\theta - yasinh\theta = ab$

$$\therefore \cancel{a}b(\cosh\theta - 1) = y\cancel{a}\sinh\theta$$

$$\therefore y = \frac{b(\cosh\theta - 1)}{\sinh\theta}$$

$$\therefore Q\left(a, \frac{b(\cosh\theta - 1)}{\sinh\theta}\right)$$

(d) Let Midpoint $= M$

$$M: \left(\frac{a + \frac{a}{\cosh\theta}}{2} , \frac{b(\cosh\theta - 1)}{2\sinh\theta} \right)$$

$$\therefore X = \frac{a\cosh\theta + a}{2\cosh\theta} \qquad y = \frac{b(\cosh\theta - 1)}{2\sinh\theta}$$

$$\text{LHS} = x(4y^2 + b^2) = \frac{a\cosh\theta + a}{2\cosh\theta}\left(4 \cdot \frac{b^2(\cosh\theta-1)^2}{4\sinh^2\theta} + b^2\right)$$

$$= \frac{\cancel{a}(a\cosh\theta \cancel{+a})}{2\cosh\theta}\left(\frac{b^2(\cosh\theta - 1)^2}{\sinh^2\theta} + b^2\right)$$

$$= \frac{ab^2(\cosh\theta+1)(\cosh\theta-1)^2}{2\cosh\theta\sinh^2\theta} + \frac{a\cosh\theta + a}{2\cosh\theta} b^2$$

$\cosh^2\theta - 1 = \sinh^2\theta$

$$= \frac{ab^2\cancel{(\cosh^2\theta - 1)}(\cosh\theta - 1)}{2\cosh\theta \cancel{\sinh^2\theta}} + b^2 \frac{a\cosh\theta + a}{2\cosh\theta}$$

$$= \frac{ab^2(\cosh\theta - 1) + b^2 a\cosh\theta + b^2 a}{2\cosh\theta}$$

$$= \frac{2ab^2 \cosh\theta}{2\cosh\theta}$$

$$= ab^2 = RHS$$

as required

$$\Rightarrow x(4y^2 + b^2) = ab^2$$